

Department of Mathematics and Computer Science
Comprehensive Examination–Option I
2014 Autumn

Algebra

- . Let $\phi : G \rightarrow G'$ be a group homomorphism with kernel K . Prove the following.
- (a) $\phi(G)$, the image of G , is abelian if and only if $xyx^{-1}y^{-1} \in K$ for all $x, y \in G$.
 - (b) $\{x \in G : \phi(x) = \phi(a)\} = Ka$ for each $a \in G$.
2. Prove that each finite integral domain is a field.
3. Let R be a ring with multiplicative identity $\neq 1$, and let F be a field. Prove that if $\phi : R \rightarrow F$ is a surjective ring homomorphism, then the kernel of ϕ is a maximal ideal in R .
4. Let V be a vector space over the field F , and let $T : V \rightarrow V$ be a linear operator on V . Prove that
- $$V_0 = \{\mathbf{v} \in V \mid T^k \mathbf{v} = \mathbf{0} \text{ for some integer } k \geq 0\}$$
- is a subspace of V , and if $T^m \mathbf{v} \in V_0$ for some $m \geq 0$, then $\mathbf{v} \in V_0$.

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Complex Analysis

1. Find the image under the transformation

$$w = \frac{z - i}{z + i}$$

of (a) $\{z \in \mathbf{C} \mid |z + 2| = 1\}$ and (b) the imaginary axis.

2. Use the method of residues to evaluate

$$\int_0^{\infty} \frac{x^2 dx}{(x^2 + 1)^3}$$

3. Find the number of zeros, counting multiplicities, of

$$f(z) = z^6 - 5z^4 + z^3 - 2z$$

inside the circle $\{z \in \mathbf{C} \mid |z| = 2\}$, and justify your conclusion.

4. Find all Laurent series expansions of

$$f(z) = \frac{1}{z(1 + z^3)}$$

centered at $z_0 = i$ and their associated regions of convergence.

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Applied Analysis

. Determine the solution $y = \phi$

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Numerical Analysis

1. (a) Prove that there exist exactly two positive solutions of the equation

$$\ln x = (x - 4)^2 - 4.$$

- (b) Find an approximation β of the smaller solution α such that $|\alpha - \beta| < 10^{-6}$.
(c) Prove that your approximation β is in fact within 10^{-6} of (the exact) α .

Note: For this problem you may not use any graphing or rootfinding capabilities of your calculator.

2. Suppose that $f^{(5)}$ is continuous. Show that

$$f'''(x_0) = \frac{-f(x_0 - 2h) + 2f(x_0 - h) - 2f(x_0 + h) + f(x_0 + 2h)}{2h^3} + O(h^2).$$

3. Let A be a $n \times n$ band matrix of the following form.

$$\begin{pmatrix} 2 & & & & & \dots \\ b_1 & 2 & & & & \dots \\ c_1 & b_2 & 2 & & & \dots \\ & c_2 & \ddots & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \ddots & c_{n-1} & b_{n-2} & \end{pmatrix}$$

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Linear Programming

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Linear Programming–continued

4. Consider the following problem.

$$\begin{array}{ll} \text{Maximize} & 5x_1 + 8x_2 + 9x_3 \\ \text{Subject to} & 2x_1 + x_2 + x_3 \leq 2 \\ & 4x_1 + 2x_2 + 3x_3 \leq 3 \\ & x_1 + 3x_2 + 3x_3 \leq 4 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Below are the first and second constraints.

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Probability

- . The negative binomial distribution is used to measure the number of Bernoulli trials one attempts until the r th success occurs. The probability mass function for the negative binomial is

$$P(X = x) =$$

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Probability–continued

3. Suppose that we toss a fair coin until a head first comes up, and let X represent the number of tosses that were made. Then the possible values of X are $1, 2, \dots$, and the distribution function of X is defined by

$$m(i) = \frac{1}{2^i}$$

which is just the geometric distribution with parameter $1/2$.

- (a) Find the expected value of X . Does this fit your intuition how it should be, given that the coin is fair?
- (b) Suppose that we flip a fair coin until a head first appears, and if the number of tosses equals n , then we are paid 2^n dollars. What is the expected value of the payment?
- (c) From what we learn in (b), how much would you be willing to pay per game for the privilege of playing this game?
4. A medical research team wishes to assess the usefulness of a certain symptom (S) in the diagnosis of a particular disease (D). In a random sample of 775 patients with the disease, 744 reported having the symptom. In an independent random sample of 38 subjects without the disease, 2 reported that they had the symptom.
- (a) Compute the sensitivity of the symptom, $P(S|D)$.
- (b) Compute the specificity of the symptom, $P(S^c|D^c)$, where c indicates the complement of the event.
- (c) Suppose it is known that the rate of the disease in the general population is $1/2$, $P(D) = 1/2$.
- i. What is the positive predictive value of the symptom, $P(D|S)$?
- ii. What is the negative predictive value of the symptom, $P(D^c|S^c)$?
- (d) Find the positive predictive values for the symptom for the following disease rates: $1/2, 1/3, 1/4, \dots$
- (e) What do you conclude about the positive predictive value of the symptom on the basis of the results obtained in part (d)?